Métodos Numéricos y Simulaciones en Astrofísica

Parte 5: Aproximaciones a las Derivadas
Método de Euler

Aproximación a la derivada mediante diferencias laterales

- Definición de derivada:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \Rightarrow f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

- Serie de Taylor:

$$f'(x) - \frac{f(x+h) - f(x)}{h} \quad = \quad f'(x) - \frac{hf'(x) + \frac{1}{2}h^2 f''(\xi_{x,h})}{h} = -\frac{1}{2}hf''(\xi_{x,h}),$$

- Entonces

$$f'(x) - \frac{f(x+h) - f(x)}{h} = -\frac{1}{2}hf''(\xi_{x,h}) = O(h).$$

- El error es aprox. proporcional a *h*.
- Tenemos algo mejor?

Aproximación a la derivada mediante diferencias centrales

- Consideremos dos series de Taylor:

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(\xi_1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(\xi_2)$$

$$\Rightarrow f(x+h) - f(x-h) = 2hf'(x) + \frac{1}{6}h^3 f'''(\xi_1) + \frac{1}{6}h^3 f'''(\xi_2)$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6}h^2 \frac{f'''(\xi_1) + f'''(\xi_2)}{2}$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6}h^2 f'''(\xi_{x,h}),$$

Aproximación a la derivada mediante diferencias

Ejemplo:

Consider, for example, the task of approximating the derivative of $f(x) = e^x$ at $x = 1$. The exact value, of course, is $f'(x) = e^x \Rightarrow f'(1) = e$. Using $h = 1/8$ and the "one-sided difference" (2.1), we get

$$f'(1) \approx \frac{e^{1.125} - e}{0.125} = 2.895480164,$$

while the same value of h with the "centered difference" (2.5) yields

$$f'(1) \approx \frac{e^{1.125} - e^{0.875}}{0.25} = 2.72536622.$$

The error in the first approximation is $-0.177...$, but the error in the second approximation is only $-7.084... \times 10^{-3}$. ■

Aproximación a la derivada mediante diferencias

- Sigamos el cálculo del ejemplo pero tomemos valores menores de h
- Hagamos que:

$$D_1(h) = \frac{f(1+h) - f(1)}{h} \qquad D_2(h) = \frac{f(1+h) - f(1-h)}{2h}$$

cuyos errores son:

$$E_1(h) = f'(1) - D_1(h) \qquad E_2(h) = f'(1) - D_2(h).$$

Aproximación a la derivada mediante diferencias

1/h	D1 (h)	E1 (h)	D2 (h)	E2 (h)
2	3.526814484	-0.808533	2.832967800	-0.114686
4	3.088244516	-0.369963	2.746685882	-0.284041E-01
8	2.895480164	-0.177198	2.725366220	-0.708439E-02
16	2.805025851	-0.867440E-01	2.720051889	-0.177006E-02
32	2.761200889	-0.429191E-01	2.718724279	-0.442450E-03
64	2.739629446	-0.213476E-01	2.718392437	-0.110609E-03
128	2.728927823	-0.106460E-01	2.718309480	-0.276519E-04
256	2.723597892	-0.531606E-02	2.718288741	-0.691295E-05
512	2.720938130	-0.265630E-02	2.718283557	-0.172824E-05
1024	2.719609547	-0.132772E-02	2.718282261	-0.432059E-06
2048	2.718945580	-0.663751E-03	2.718281936	-0.108015E-06
4096	2.718613677	-0.331849E-03	2.718281855	-0.270040E-07
8192	2.718447746	-0.165918E-03	2.718281835	-0.675060E-08
16384	2.718364786	-0.829571E-04	2.718281830	-0.168773E-08
32768	2.718323307	-0.414781E-04	2.718281829	-0.421277E-09
65536	2.718302567	-0.207390E-04	2.718281829	-0.111843E-09
131072	2.718292198	-0.103695E-04	2.718281828	-0.185980E-10
262144	2.718287013	-0.518476E-05	2.718281828	-0.183991E-10
524288	2.718284421	-0.259243E-05	2.718281828	-0.654721E-11
1048576	2.718283125	-0.129633E-05	2.718281828	0.272178E-10
2097152	2.718282477	-0.648392E-06	2.718281829	-0.739635E-10
4194304	2.718282153	-0.324651E-06	2.718281829	-0.359772E-09

Aproximación a la derivada mediante diferencias

1/h	D2 (h)	E2 (h)	E2 (h/2)/E2 (h)
2	2.832967800	-0.114686	
4	2.746685882	-0.284041E-01	4.038
8	2.725366220	-0.708439E-02	4.009
16	2.720051889	-0.177006E-02	4.002
32	2.718724279	-0.442450E-03	4.001
64	2.718392437	-0.110609E-03	4.000
128	2.718309480	-0.276519E-04	4.000
256	2.718288741	-0.691295E-05	4.000
512	2.718283557	-0.172824E-05	4.000
1024	2.718282261	-0.432059E-06	4.000
2048	2.718281936	-0.108015E-06	4.000
4096	2.718281855	-0.270040E-07	4.000
8192	2.718281835	-0.675060E-08	4.000
16384	2.718281830	-0.168773E-08	4.000
32768	2.718281829	-0.421277E-09	4.006
65536	2.718281829	-0.111843E-09	3.767
131072	2.718281828	-0.185980E-10	6.014
262144	2.718281828	-0.183991E-10	1.011
524288	2.718281828	-0.654721E-11	2.810
1048576	2.718281828	0.272178E-10	-0.241
2097152	2.718281829	-0.739635E-10	-0.368
4194304	2.718281829	-0.359772E-09	0.206

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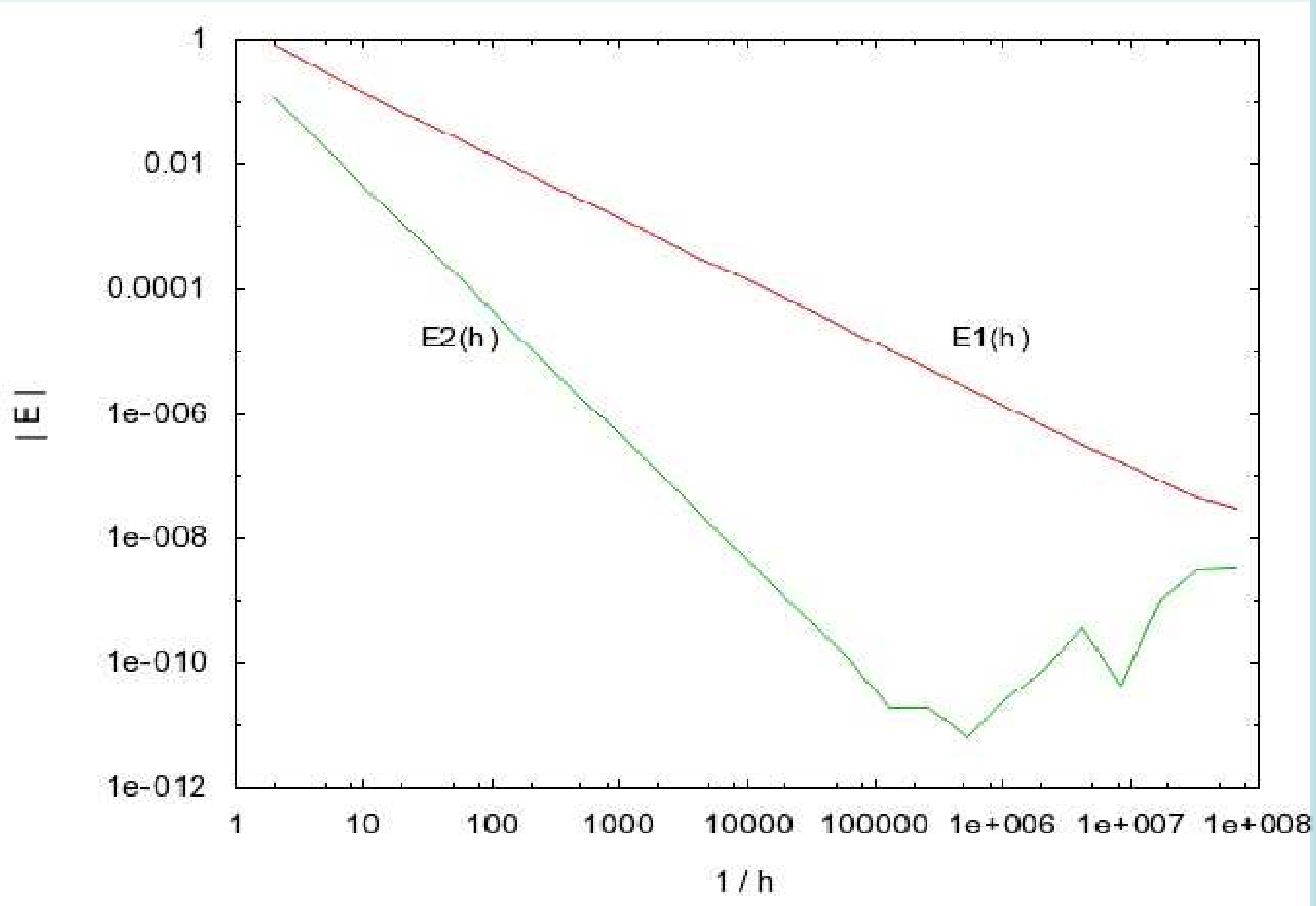
???

El error Aumenta!

Aproximación a la derivada mediante diferencias



Error de Redondeo

- Si indicamos con $\tilde{f}(x)$ la función calculada por la máquina.
- Definimos $\epsilon(x) = f(x) - \tilde{f}(x)$ como el error entre la función calculada con precisión infinita y la que utiliza la máquina.
- La derivada aproximada que se calcula se construye con la función calculada, no con *f*.
- Definamos:

$$\tilde{D}_2(h) = \frac{\tilde{f}(x+h) - \tilde{f}(x-h)}{2h}$$

Error de Redondeo

- Tenemos que:

$$f'(x) - \tilde{D}_2(h) = f'(x) - \frac{\tilde{f}(x+h) - \tilde{f}(x-h)}{2h}$$

que lo escribimos como:

$$\begin{aligned}
f'(x) - \tilde{D}_2(h) &= f'(x) - \frac{f(x+h) - f(x-h)}{2h} + \frac{f(x+h) - f(x-h)}{2h} - \frac{\tilde{f}(x+h) - \tilde{f}(x-h)}{2h} \\
&= -\frac{1}{6}h^2 f'''(\xi_{x,h}) + \frac{f(x+h) - f(x-h) - \tilde{f}(x+h) + \tilde{f}(x-h)}{2h} \\
&= \underbrace{-\frac{1}{6}h^2 f'''(\xi_{x,h})}_{\text{Error due to approximation}} + \underbrace{\frac{\epsilon(x+h) - \epsilon(x-h)}{2h}}_{\text{Error due to rounding}}.
\end{aligned}$$

Método de Euler para problemas de valor inicial

- Forma general:

$$y' = f(t,y), \quad y(t_0) = y_0,$$

where f is a known function of t and y, and t_0, y_0 are given values.

- Diferencia lateral:

$$\frac{y(t+h) - y(t)}{h} = f(t,y(t)) + \frac{1}{2}hy''(t_h)$$

$$y(t+h) = y(t) + hf(t,y(t)) + \frac{1}{2}h^2y''(t_h)$$

- Método de Euler:

1. Define a sequence of t values (called a *grid*) according to $t_n = t_0 + nh$, where h is a set parameter (called the *mesh spacing* or *grid size*; we will encounter this kind of thing often in later topics).
2. Compute the values y_n from y_0 and the t grid values, according to

$$y_{n+1} = y_n + hf(t_n,y_n). \tag{2.10}$$

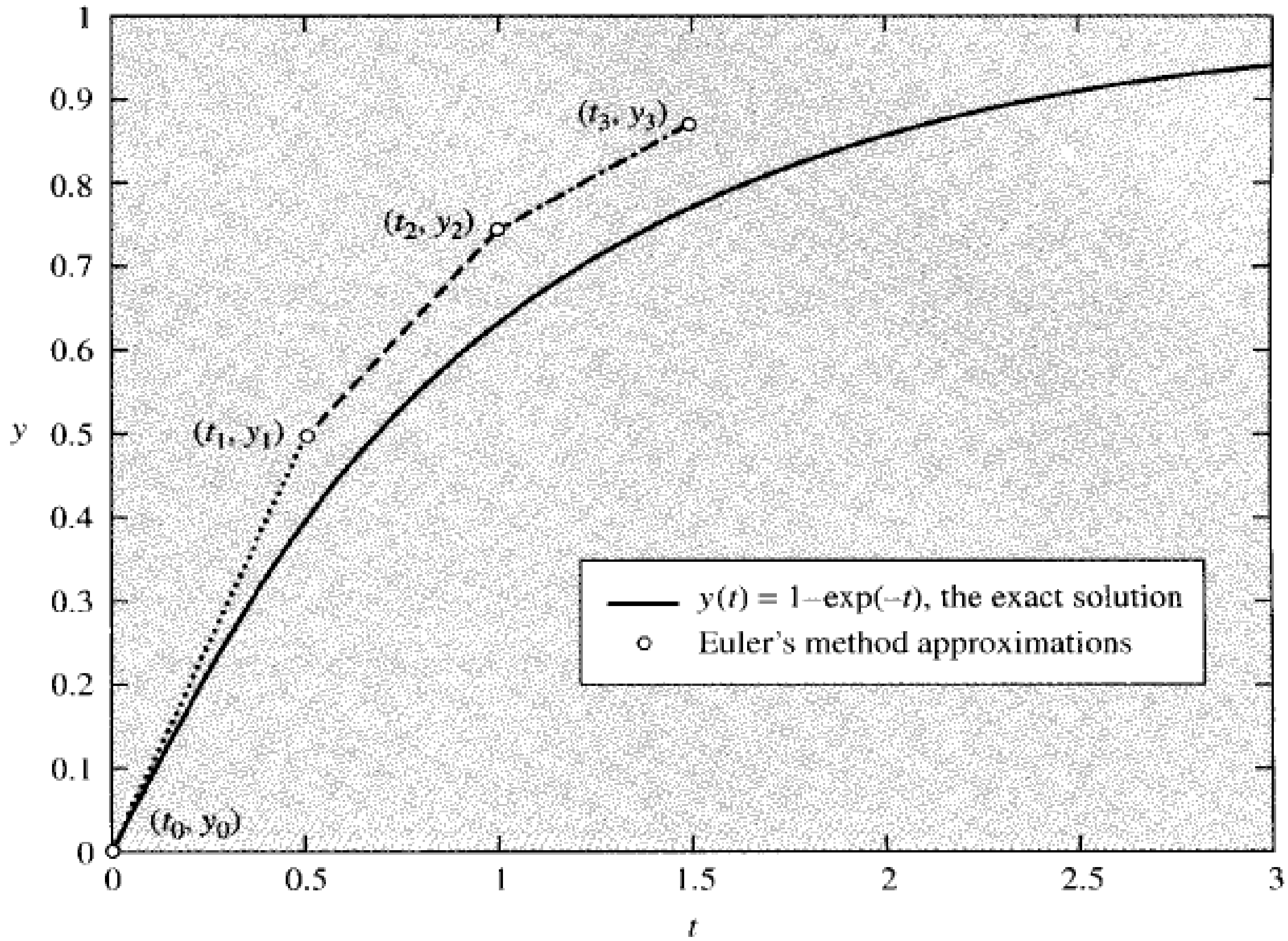


FIGURE 2.3 Geometric derivation of Euler's Method.

Método de Euler para problemas de valor inicial

Consider the very simple initial value problem

$$y' = -y + \sin t, \quad y(0) = 1.$$

This has exact solution $y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}(\sin t - \cos t)$, found by using the kinds of methods taught in the usual ODE course. If we apply Euler's method to this, using $h = \frac{1}{4}$, then we get the following results.

Step 1: We have $h = \frac{1}{4}$, so $t_1 = h = \frac{1}{4}$, and y_0 is given as 1. Then

$$y_1 = y_0 + hf(t_0, y_0) = 1 + \frac{1}{4}(-1 + \sin 0) = \frac{3}{4}.$$

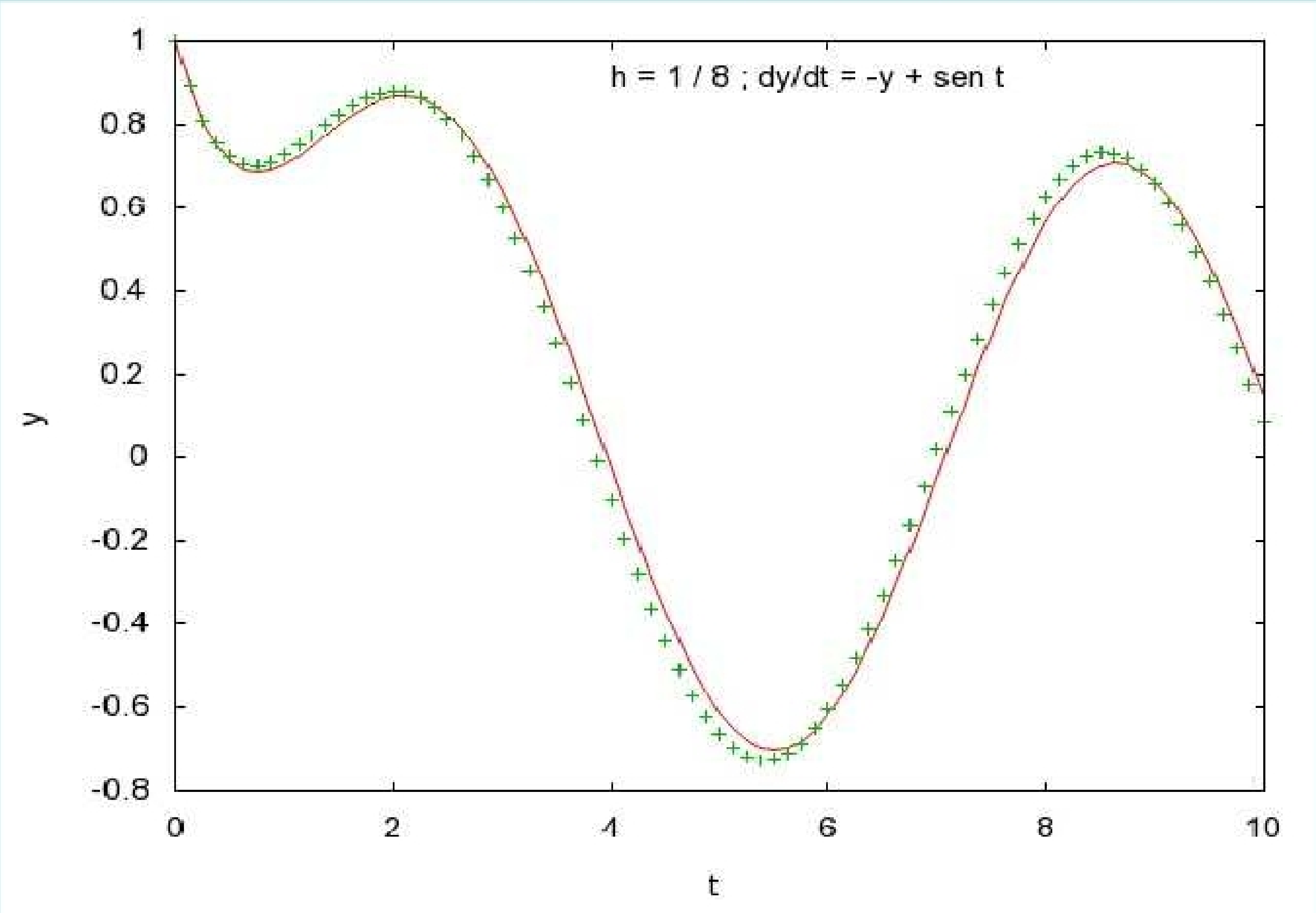
Thus $y(1/4) \approx 0.75$, and the error in this approximation is $e_1 = y(1/4) - y_1 = 0.8074469434 - 0.75 = 0.0574469434$.

Step 2: We have $t_2 = 2h = \frac{1}{2}$ and $y_1 = 0.75$ from the previous step. Then

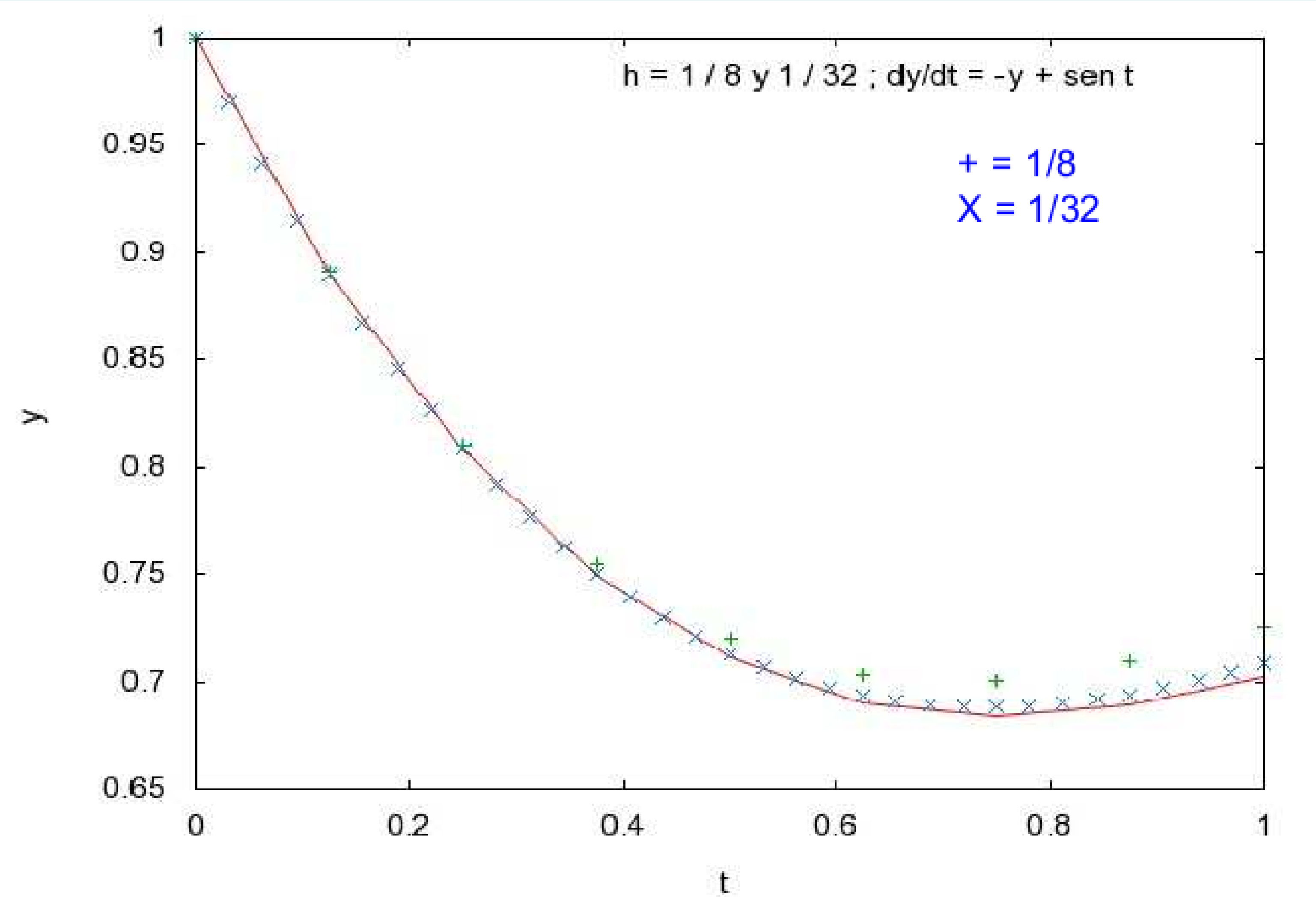
$$y_2 = y_1 + hf(t_1, y_1) = 3/4 + \frac{1}{4}\left(-3/4 + \sin\frac{1}{4}\right) = 0.6243509898.$$

Thus $y(1/2) \approx 0.6243509898$, and the error in this approximation is $e_2 = y(1/2) - y_2 = 0.7107174779 - 0.6243509898 = 0.0863664881$. ■

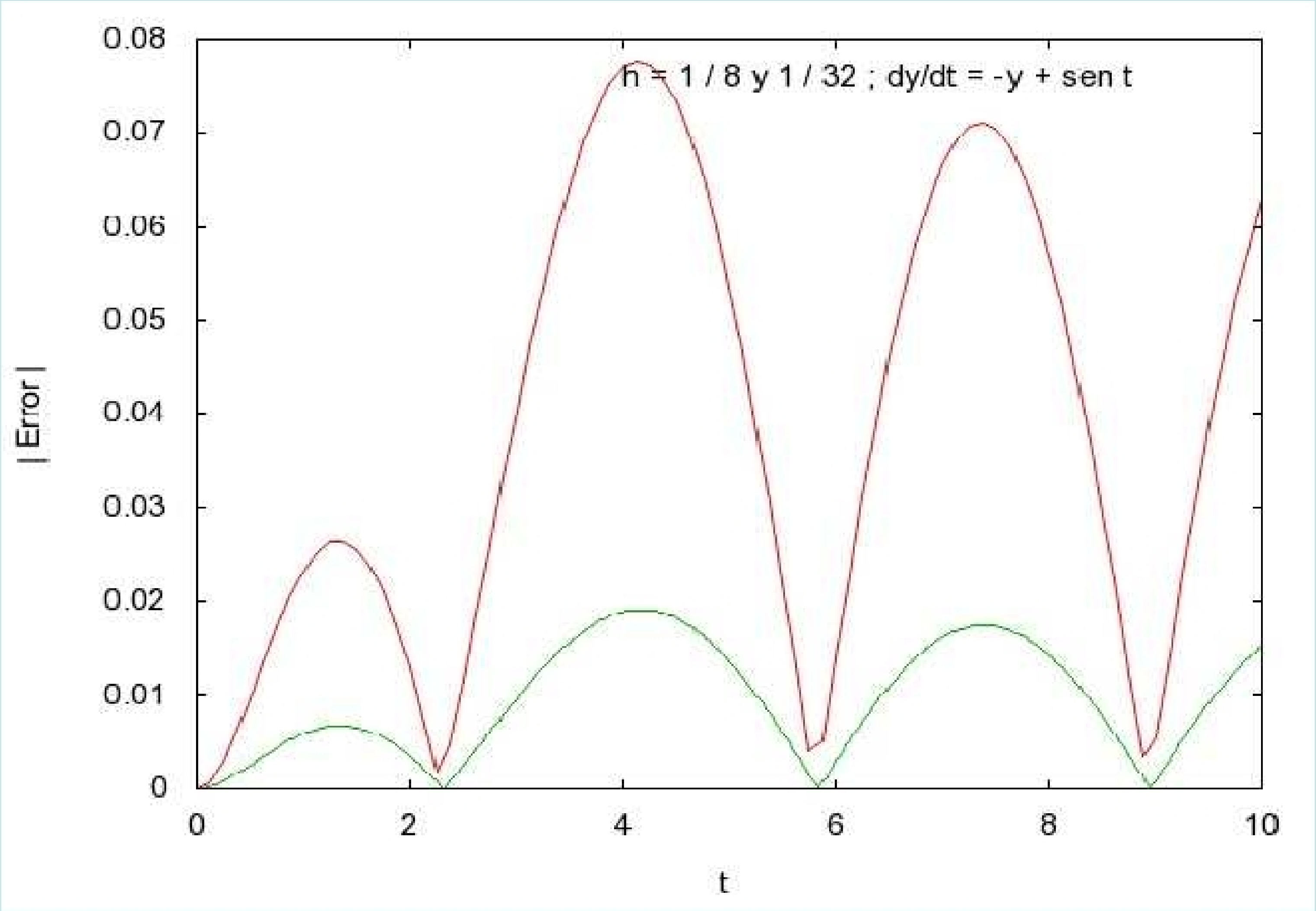
Método de Euler para problemas de valor inicial



Método de Euler para problemas de valor inicial



Método de Euler para problemas de valor inicial



Método de Euler para problemas de valor inicial

Consideremos otro problema de valor inicial:

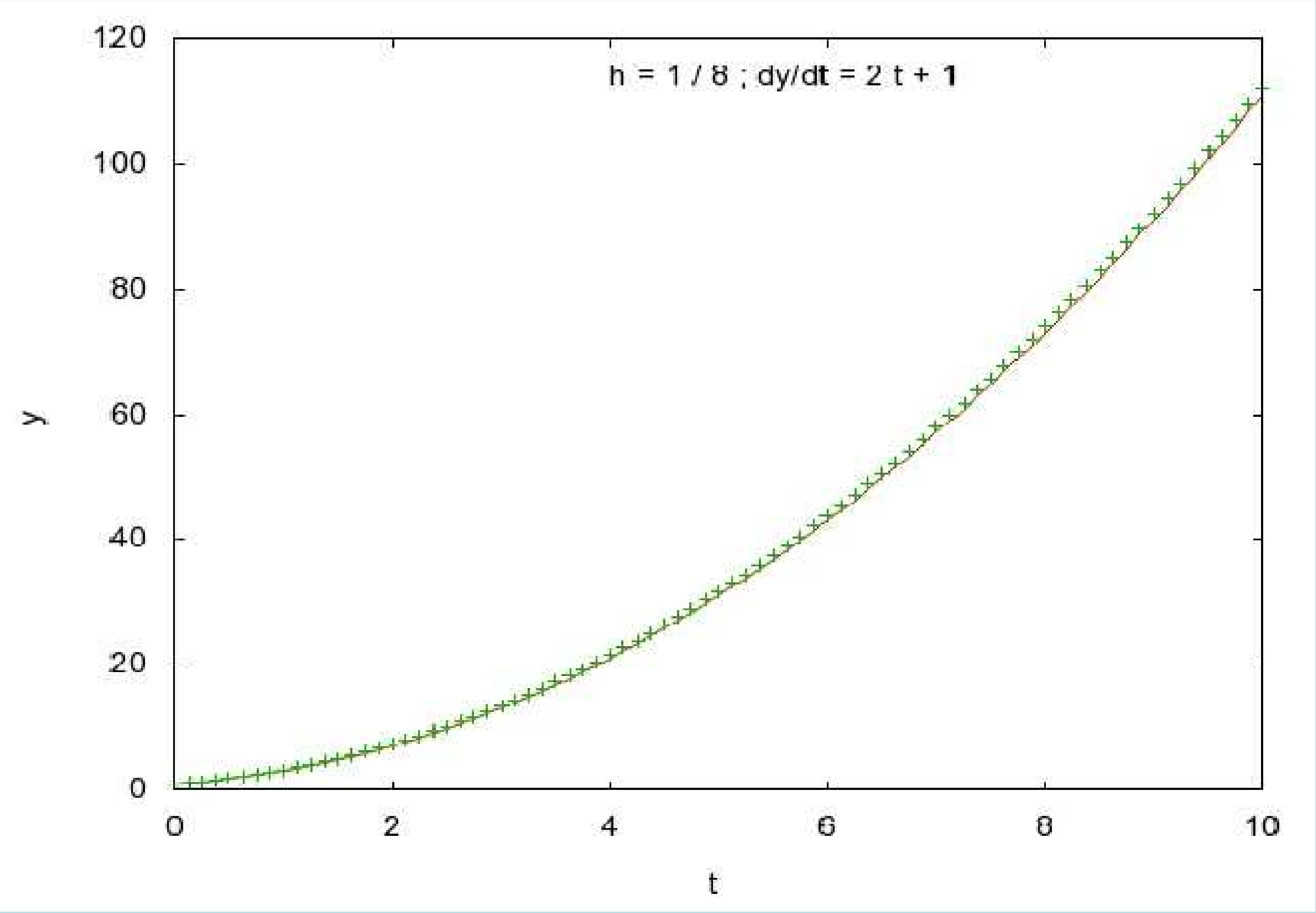
$$dy/dt = 2\,t + 1 \quad \text{con } y(0) = 1$$

Cuya solución es:

$$y = t^2 + t + 1$$

Y utilicemos el método de Euler nuevamente.

Método de Euler para problemas de valor inicial



Método de Euler para problemas de valor inicial