Métodos Numéricos y Simulaciones en Astrofísica

Parte 6: Ecuaciones Diferenciales

Criterios

- Para resolver ecuaciones diferenciales tenemos que considerar que:
  - Si la función f es “suficientemente suave”, entonces:
    - Existe una solución y es única
    - Se podrá aproximar lo suficientemente bien aplicando algún método de cálculo
  - Para evaluar si f es “suficientemente suave”:
    - Continuidad de Lipschitz
    - Suave y uniformemente monótona decreciente

Criterios

Definition 6.1 (Lipschitz Continuity) *Let g be a given function from* **R** *to* **R**. *We say that g is* Lipschitz[2] continuous *on an interval I if there exists a constant K such that*

$$|g(x_1) - g(x_2)| \leq K|x_1 - x_2|$$

for all $x_1, x_2 \in I$.

Definition 6.2 (Smooth and Uniformly Monotone Decreasing) *Let g be a given function from* **R** *to* **R**. *We say that g is* smooth and uniformly monotone decreasing *if g is differentiable and the derivative satisfies*

$$-M \leq g'(x) \leq -m < 0$$

for all x, where M and m are given positive constants.

Criterios

Ejemplo:

Just to illustrate, consider the two initial value problems

$$y' = 4y - e^{-t}, \quad y(0) = 1 \tag{6.5}$$

and

$$y' = -(1+t^2)y + \sin t, \quad y(0) = 1. \tag{6.6}$$

For (6.5) we have $f(t,y) = 4y - e^{-t}$ so that

$$f(t,y_1) - f(t,y_2) = (4y_1 - e^{-t}) - (4y_2 - e^{-t}) = 4(y_1 - y_2);$$

hence

$$|f(t,y_1) - f(t,y_2)| \leq 4|y_1 - y_2|.$$

Hence, in this case f is Lipschitz continuous in y with constant $K = 4$. However, f is not smooth and uniformly monotone decreasing, because $f_y(t,y) = 4 > 0$ for all t and y.

Criterios

Ejemplo:

$$y' = -(1+t^2)y + \sin t, \quad y(0) = 1. \tag{6.6}$$

On the other hand, for (6.6) we have $f(t,y) = -(1+t^2)y + \sin t$, so

$$f(t,y_1) - f(t,y_2) = (-(t^2+1)y_1 + \sin t) - (-(t^2+1)y_2 + \sin t) = -(t^2+1)(y_1 - y_2);$$

thus, for $0 \le t \le 1$,

$$|f(t,y_1) - f(t,y_2)| \le (t^2+1)|y_1 - y_2| \le 2|y_1 - y_2|.$$

Therefore, this f is Lipschitz continuous with constant $K = 2$. In addition, we have (again, for $0 \le t \le 1$)

$$f_y(t,y) = -(t^2+1) \quad \Rightarrow \quad -2 \le f_y(t,y) \le -1 < 0,$$

showing that this f is also smooth and uniformly monotone decreasing. ∎

Criterios

Existencia y unicidad:

Theorem 6.1 (Existence-Uniqueness of Solutions for IVPs, Version I) *Let $f(t,y)$ be continuous for all (t,y) in an open rectangle $R = \{(t,y) : a < t < b, c < y < d\}$ and Lipschitz continuous in y, with constant K. Then for all $(t_0,y_0) \in R$ there exists a unique solution to the initial value problem*

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Moreover, if $z(t)$ is the solution to the same problem with initial data $z(t_0) = z_0$, then

$$|y(t) - z(t)| \leq e^{K(t-t_0)}|y_0 - z_0|. \tag{6.7}$$

Método de Euler

- Planteo analítico (Taylor):

$$y(t+h) = y(t) + hf(t, y(t)) + \frac{1}{2}h^2 y''(\theta).$$

- Término residual:

$$R(t,h) = \frac{1}{2}h^2 y''(\theta)$$

- Error de truncamiento:

$$\tau(t,h) = \frac{1}{h}R(t,h)$$

Método de Euler

Estimación del error:

Theorem 6.3 (Error Estimate for Euler's Method, Version I) *Let f be Lipschitz continuous, with constant K, and assume that the solution $y \in C^2([t_0,T])$ for some $T > t_0$. Then*

$$\max_{t_k \le T} |y(t_k) - y_k| \le C_0 |y(t_0) - y_0| + Ch \|y''\|_{\infty,[t_0,T]},$$

O(h)

where

$$C_0 = e^{K(T-t_0)}$$

and

$$C = \frac{e^{K(T-t_0)} - 1}{2K}. \qquad (6.16)$$

Método de Euler

Estimación del error:

Theorem 6.4 (*Error Estimate for Euler's Method, Version II*)

Let f be smooth and uniformly monotone decreasing in y, and assume that the solution $y \in C^2([t_0, T])$ for some $T > t_0$.

Then for h sufficiently small,

$$\max_{t_k \le T} |y(t_k) - y_k| \le C_0 |y(t_0) - y_0| + Ch \|y''\|_{\infty,[t_0,T]}$$

O(h)

where $C_0 \le 1, C_0 \to 0$ as $k \to \infty$, and

$$C = \frac{1}{2m}$$

Método de Euler

Estimación del error:

Ambos teoremas muestran que el método de Euler es de primer orden ($O(h)$), y que:

- Si f es contínua de Lipschitz, entonces el producto de la constante por el error inicial y el paso pueden resultan en valores **grandes y rápidamente crecientes**.
- Si f es suave y uniformemente monótona decreciente, las constantes en la estimación del error están **acotadas para todo n**.

Método de Euler

Estimación del error:

Cómo se ve afectado el error inicial por f?

- Si f es monótona decreciente, el efecto del error inicial **decrece rápidamente**.
- Si f es solo contínua de Lipschitz, cualquier error inicial puede ser **amplificado en forma exponencial**.

Variantes del Método de Euler

Ecuación diferencial: $y'(t) = f(t, y(t))$

Método de Euler:

$$\frac{y(t+h) - y(t)}{h} = f(t, y(t)) + \frac{1}{2} h y''(\theta_{t,h}),$$ O(h)

$$y_{n+1} = y_n + h f(t_n, y_n).$$

1. Euler hacia atrás:

$$y'(t) = \frac{y(t) - y(t-h)}{h} - \frac{1}{2} h y''(\theta),$$ O(h)

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) :$$

2. Punto intermedio:

$$y'(t) = \frac{y(t+h) - y(t-h)}{2h} - \frac{1}{6} h^2 y'''(\theta_{t,h}),$$ O(h^2)

$$y_{n+1} = y_{n-1} + 2h f(t_n, y_n).$$

Variantes del Método de Euler

Métodos de interpolación:

1. Given a set of *nodes* $\{x_i, 0 \le i \le n\}$ and corresponding data values $\{y_i, 0 \le i \le n\}$, find the polynomial $p_n(x)$ of degree less than or equal to n, such that

$$p_n(x_i) = y_i, \qquad 0 \le i \le n.$$

2. Given a set of *nodes* $\{x_i, 0 \le i \le n\}$ and a continuous function $f(x)$, find the polynomial $p_n(x)$ of degree less than or equal to n, such that

$$p_n(x_i) = f(x_i), \qquad 0 \le i \le n.$$

Variantes del Método de Euler

Métodos de interpolación:

Definimos:

$$L_i^{(n)}(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{x - x_k}{x_i - x_k}$$

Entonces:

$$p_n(x) = \sum_{k=0}^{n} y_k L_k^{(n)}(x),$$

$$p_n(x_i) = \sum_{k=0}^{n} y_k L_k^{(n)}(x_i) = y_i.$$

Polinomio de Lagrange

Variantes del Método de Euler

Métodos de interpolación:

Theorem 4.3 (Polynomial Interpolation Error Theorem) Let $f \in C^{n+1}([a,b])$ and let the nodes $x_k \in [a,b]$ for $0 \leq k \leq n$. Then, for each $x \in [a,b]$, there is a $\xi_x \in [a,b]$ such that

$$f(x) - p_n(x) = \frac{w_n(x)}{(n+1)!} f^{(n+1)}(\xi_x), \qquad (4.6)$$

where

$$w_n(x) = \prod_{k=0}^{n} (x - x_k).$$

Variantes del Método de Euler

Métodos de interpolación:

The interpolation error theorem gives us

$$f(x) - p_n(x) = \frac{1}{(n+1)!} w_n(x) f^{(n+1)}(\xi_x),$$

and the fact that this is an equality is important, for we can now differentiate both sides to get

$$f'(x) - p_n'(x) = \frac{1}{(n+1)!} \frac{d}{dx} \left[w_n(x) f^{(n+1)}(\xi_x) \right].$$

$$\frac{d}{dx} \left[w_n(x) f^{(n+1)}(\xi_x) \right] = w_n'(x) f^{(n+1)}(\xi_x) + w_n(x) \frac{d}{dx} \left[f^{(n+1)}(\xi_x) \right].$$

$$f'(x_i) - p_n'(x_i) = \frac{1}{(n+1)!} w_n'(x_i) f^{(n+1)}(\xi_i),$$

Variantes del Método de Euler

Métodos de interpolación:

$$p_n(x) = \sum_{k=0}^{n} y_k L_k^{(n)}(x),$$

$$p_2(x) = f(x_0)L_0^{(2)}(x) + f(x_1)L_1^{(2)}(x) + f(x_2)L_2^{(2)}(x).$$

$$f'(x_i) \approx p_2'(x_i) = f(x_0)(L_0^{(2)})'(x_i) + f(x_1)(L_1^{(2)})'(x_i) + f(x_2)(L_2^{(2)})'(x_i),$$

$$f'(x_i) - p_2'(x_i) = \frac{1}{6} w_2'(x_i) f'''(\xi_i)$$

$$(L_0^{(2)})'(x_0) = -3/2h \qquad (L_1^{(2)})'(x_0) = 2/h \qquad (L_2^{(2)})'(x_0) = -1/2h$$

$$(L_0^{(2)})'(x_1) = -1/2h \qquad (L_1^{(2)})'(x_1) = 0 \qquad (L_2^{(2)})'(x_1) = 1/2h$$

$$(L_0^{(2)})'(x_2) = 1/2h \qquad (L_1^{(2)})'(x_2) = -2/h \qquad (L_2^{(2)})'(x_2) = 3/2h$$

and

$$w_2'(x_0) = 2h^2 \qquad w_2'(x_1) = -h^2 \qquad w_2'(x_2) = 2h^2$$

Variantes del Método de Euler

Métodos de interpolación:

$$f'(x_i) \approx p_2'(x_i) = f(x_0)(L_0^{(2)})'(x_i) + f(x_1)(L_1^{(2)})'(x_i) + f(x_2)(L_2^{(2)})'(x_i),$$

so that the approximations are

$$\begin{aligned} f'(x_0) &\approx \frac{1}{2h}(-f(x_2) + 4f(x_1) - 3f(x_0)) \\ f'(x_1) &\approx \frac{1}{2h}(f(x_2) - f(x_0)) \\ f'(x_2) &\approx \frac{1}{2h}(3f(x_2) - 4f(x_1) + f(x_0)) \end{aligned}$$

with errors

$$f'(x_0) - \frac{1}{2h}(-f(x_2) + 4f(x_1) - 3f(x_0)) = \frac{1}{3}h^2 f'''(\xi_0)$$

$$f'(x_1) - \frac{1}{2h}(f(x_2) - f(x_0)) = -\frac{1}{6}h^2 f'''(\xi_1)$$

$$f'(x_2) - \frac{1}{2h}(3f(x_2) - 4f(x_1) + f(x_0)) = \frac{1}{3}h^2 f'''(\xi_2).$$

$O(h^2)$

Variantes del Método de Euler

Métodos de interpolación:

$$f'(x) = \left(\frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} \right) + O(h^2)$$

$$f'(x) = \left(\frac{3f(x) - 4f(x-h) + f(x-2h)}{2h} \right) + O(h^2).$$

$$y'(t) \approx \frac{1}{2h}\left(-y(t+2h) + 4y(t+h) - 3y(t)\right)$$

$$y'(t+2h) \approx \frac{1}{2h}\left(3y(t+2h) - 4y(t+h) + y(t)\right),$$

Variantes del Método de Euler

Métodos de interpolación:

3. Método interpolar:

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(t_{n-1}, y_{n-1})$$

4. Método interpolar:

$$y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1} + \frac{2}{3}hf(t_{n+1}, y_{n+1}).$$

Variantes del Método de Euler

Método de integración:

Suma de Riemann:

$$R_n(f) = \sum_{i=1}^{n} f(\eta_i)(x_i^{(n)} - x_{i-1}^{(n)})$$

Regla del trapecio:

$$T_n(f) = \frac{h}{2}(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n))$$

$$I(f) - T_n(f) = -\frac{b-a}{12} h^2 f''(\xi_h),$$

$$I(f) - T_n(f) = -\frac{1}{12} h^3 \sum_{i=1}^{n} f''(\xi_{i,h}).$$

$$h^3 \sum_{i=1}^{n} f''(\xi_{i,h}) = h^2 \sum_{i=1}^{n} h f''(\xi_{i,h}),$$

$$I(f'') = \int_a^b f''(x)\, dx = f'(b) - f'(a).$$

Variantes del Método de Euler

Método de integración:

Ecuación diferencial:

$$y'(t) = f(t, y(t))$$

La integro en (t,t+h):

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))\, ds.$$

Aplico regla trapecio:

$$y(t+h) = y(t) + \frac{1}{2}h[f(t+h, y(t+h)) + f(t, y(t))] - \frac{1}{12}h^3 y'''(\theta_{t,h}),$$

O(h³)

5. Método de integración trapezoidal:

$$y_{n+1} = y_n + \frac{1}{2}h\left(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)\right).$$

Variantes del Método de Euler

Método de integración:

Se puede aplicar una regla de punto intermedio para integrar:

$$y(t+h) = y(t) + hf\left(t + \frac{1}{2}h, y\left(t + \frac{1}{2}h\right)\right) - \frac{1}{24}h^3 y'''(\theta_{t,h}),$$

$O(h^3)$

6. Método de integración de punto medio:

$$y_{n+1} = y_n + hf(t_{n+1/2}, y_{n+1/2}),$$

where $t_{n+1/2} = t_n + \frac{1}{2}h$ and $y_{n+1/2} \approx y(t_n + \frac{1}{2}h)$.

Variantes del Método de Euler

Análisis:

- Los métodos 2 a 6 son de orden superior al Euler.
- Los métodos 2 a 4 son métodos de **paso múltiple** porque dependen de aproximaciones a más de un valor de la función desconocida.
- Los métodos 1, 4 y 5 dependen de $f(t_{n+1}, y_{n+1})$ por lo cual no se puede encontrar directamente una aproximación y_{n+1}. Estos son **métodos implícitos**, mientras que el 2 y 3 son **métodos explícitos**.

Residuo y error de truncamiento

We are, of course, interested in the accuracy of the methods we develop here. All of the numerical methods for IVPs that we will study can be written in the general format

$$y_{n+1} = \sum_{k=0}^{p} a_k y_{n-k} + hF\left(y_{n+1}, y_n, \ldots, y_{n-p}; f_{n+1}, f_n, \ldots, f_{n-p}\right) \qquad (6.27)$$

where we have used $f_k = f(t_k, y_k)$ for simplicity. Thus, for example, in Euler's method we have

$$y_{n+1} = y_n + hf(t_n, y_n)$$

so that

$$a_0 = 1$$
$$a_k = 0, \quad \text{all } k \geq 1,$$

and

$$F\left(y_{n+1}, y_n, \ldots, y_{n-p}; f_{n+1}, f_n, \ldots, f_{n-p}\right) = f_n = f(t_n, y_n).$$

Residuo, truncamiento y consistencia

Definition 6.3 (Residual, Truncation Error, and Consistency) *For a numerical method written as in (6.27), define the* residual *as R_n, given by*

$$R_n = y(t_{n+1}) - \sum_{k=0}^{p} a_k y(t_{n-k}) - hF\left(y(t_{n+1}), y(t_n), \ldots, y(t_{n-p}); f(t_{n+1}, y(t_{n+1})), f(t_n, y(t_n)), \ldots, f(t_{n-p}, y(t_{n-p}))\right).$$

The truncation error *is defined as*

$$\tau_n = \frac{1}{h} R_n,$$

and the method is said to be consistent *if*

$$\lim_{h \to 0} \max_{t_n \le T} |\tau_n| = 0$$

for sufficiently smooth solutions y.

Residuo, truncamiento y consistencia

Ejemplo:

Using what we know of the trapezoid rule for approximating integrals, we can write (see Section 2.5)

$$y(t_{n+1}) = y(t_n) + \frac{1}{2}h\left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right) - \frac{1}{12}h^3 y'''(\theta_n);$$

therefore, the residual is $-\frac{1}{12}h^3 y'''(\theta_n)$, and the truncation error is $-\frac{1}{12}h^2 y'''(\theta_n)$. This time, the method is consistent if y is C^3. ∎

Estabilidad

Si Y_n es el valor real de Y en la n-esima iteración:

$$Y_{n+1} = Y_n + hf(t_n, Y_n)$$

Pero se utilizan valores calculados $y_{n:}$

$$y_{n+1} = y_n + hf(t_n, y_n)$$

Donde:

$$y_n = Y_n + \epsilon_n$$

Entonces:

$$(Y_{n+1} + \epsilon_{n+1}) = (Y_n + \epsilon_n) + hf(t_n, Y_n + \epsilon_n)$$

Estabilidad

Restando la primera expresión:

$$\epsilon_{n+1} = \epsilon_n + h[f(t_n, Y_n + \epsilon_n) - f(t_n, Y_n)]$$

O lo que es lo mismo:

$$\epsilon_{n+1} = \epsilon_n + h\epsilon_n \left[\frac{\partial f}{\partial y}\right]_{y=Y_n} = \epsilon_n \left(1 + h \left[\frac{\partial f}{\partial y}\right]_{y=Y_n}\right)$$

Generalizando:

$$\epsilon_{n+1} = \epsilon_0 \left(1 + h \left[\frac{\partial f}{\partial y}\right]_{y=Y_n}\right)^n$$

Estabilidad

Entonces, si se asume una solución oscilante podemos escribir:

$$\left| 1 + h \left[\frac{\partial f}{\partial y} \right]_{y=Y_n} \right| < \left| 1 - h \left[\frac{\partial f}{\partial y} \right]_{max} \right| \leq 1$$

Lo que permite fijar h:

$$h < \frac{1}{\left| \left[\frac{\partial f}{\partial y} \right]_{max} \right|}$$