Coarse geometry and Roe C*-algebras

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Outline

Objects of Interest

Slogan: Study spaces from a large–scale perspective.

Notation: $X, Y \ldots$ metric spaces; d metric. Γ discrete group.

Examples:

- Finitely generated groups with word metric: $\Gamma = \langle S \rangle$, $S = S^{-1}$, $|S| < \infty$. Define a metric by $d_S(g,h) =$ the length of a shortest word in alphabet S representing $g^{-1}h$.
  E.g. if $\mathbb{Z} = \langle 1, -1 \rangle$, then the metric is $d(m,n) = |n-m|$.
- Graphs (finite or infinite), endowed with the path metric.
- Complete Riemannian manifolds.
- "Coarse disjoint union" $X = \sqcup_n G_n$ of a sequence of finite graphs (G_n). Metric: on each G_n the path metric, $d(G_n, G_m) = m + n + |G_n| + |G_m|$.
- "Box space": $X = \sqcup_n \mathrm{Cayley}(\Gamma/\Gamma_n; S/\Gamma_n)$, where $\Gamma = \langle S \rangle$, $|S| < \infty$, $\Gamma_n \leq \Gamma$ normal, $[\Gamma : \Gamma_n] < \infty$.

Expanders

For a finite graph G, the Cheeger constant is

$$h(G) = \min \left\{ \frac{|\partial S|}{|S|} : S \subset V(G), 0 < |S| \leq \frac{|G_n|}{2} \right\}.$$

A *sequence of expanders* (expander) is a sequence of finite graphs G_n, such that

- the degrees of vertices are uniformly bounded,
- $|G_n| \nearrow \infty$ and
- $\inf_n h(G_n) > 0$.

Think of it as a metric space $X = \sqcup_n G_n$.

First examples: Box spaces of residually finite groups with property (T), e.g. $\sqcup_p SL_n(\mathbb{Z})/SL_n(\mathbb{Z}/p\mathbb{Z})$. [Margulis]

In fact, a group Γ has property (τ) with respect to a family of finite index subgroups $(\Gamma_n)_{n \in \mathbb{N}}$ iff the box space $X = \sqcup_n (\Gamma/\Gamma_n)$ is an expander.

Coarse Definitions

A map $f : X \to Z$ is *coarse*, if

- $\exists \rho_+ : [0,\infty) \to [0,\infty)$, such that $d(f(x),f(y)) \le \rho_+(d(x,y))$ for all $x,y \in X$;
- $|f^{-1}(z)| < \infty$ for all $z \in Z$.

Maps $f,g : X \to Z$ are *close*, if $\sup\{d(f(x),g(x)) \mid x \in X\} < \infty$.
Spaces X and Z are *coarsely equivalent*, written $X \sim_c Z$, if there exist coarse $f : X \to Z$, $g : Z \to X$, with $f \circ g$ is close to id_Z and $g \circ f$ is close to id_X.
A map $f : X \to Z$ is a $(\star)$ *embedding*, if there exist $\rho_+, \rho_- : [0,\infty) \to [0,\infty)$ with $(\star\star)$, such that

$$\rho_-(d(x,y)) \le d(f(x),f(y)) \le \rho_+(d(x,y)).$$

$(\star)$	$(\star\star)$
coarse (CE)	$\rho_- \nearrow \infty$
quasi–isometric (QI)	ρ_+ and ρ_- are affine $(Ax+B)$
bilipschitz	ρ_+ and ρ_- are linear (Ax)

A $(\star)$ embedding f is a $(\star)$ equivalence, if $\sup\{d(z,f(X)) \mid z \in Z\} < \infty$.

Examples

Equivalences:

- Bounded space $\sim_c \{\mathrm{pt}\}$.
- $\mathbb{Z}^n \sim_c \mathbb{R}^n$, but $\mathbb{Z}^m \sim_c \mathbb{Z}^n \implies m = n$ (asymptotic dimension).
- Free groups F_r (with the free generating sets), $2 \le r < \infty$ are all bilipschitz equivalent [Papasoglu '95].
- $X = \{2^{2^n} \mid n \in \mathbb{N}\} \subset \mathbb{N}$, metric from $\mathbb{N}$. Any bijection is a coarse equivalence, but any QI $X \to X$ is eventually constant.

Basic Geometric Group Theory Lemma:
If $\Gamma = \langle S \rangle = \langle S' \rangle$, $|S|, |S'| < \infty$, then $(\Gamma, d_S) \sim_{QI} (\Gamma, d_{S'})$. So any QI invariant is actually an invariant of the underlying group.

Švarc–Milnor Theorem:
If Γ acts properly and cocompactly on a length space X, then $\Gamma \sim_{QI} X$.
[So $\pi_1(M) \sim_{QI} \widetilde{M}$ for a compact Riemannian manifold M.]

Coarse Properties and Some Theorems

...amenability, asymptotic dimension, coarse embeddability into ___ (e.g. a Hilbert space)

(Gromov) hyperbolicity is a QI invariant of geodesic spaces.

Theorem (G. Yu '97)

If M is a uniformly contractible complete Riemannian manifold with bounded geometry and finite asymptotic dimension, then it admits no metric of uniformly positive scalar curvature, within the class of CE metrics.

Theorem (G. Yu '00)

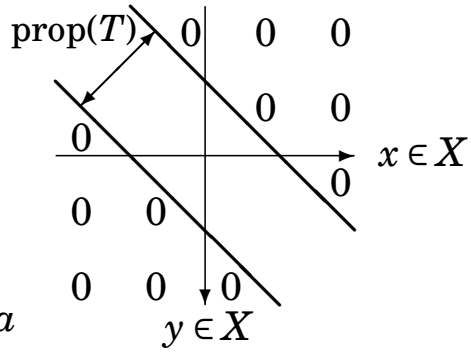
If Γ admits a coarse embedding into a Hilbert space, then the Novikov conjecture holds for Γ.

Definitions of Roe algebras

Slogan: Roe algebras are C*-algebras which encode the coarse structure of a space.

Let X be a uniformly discrete [$\exists c > 0$ with $x \neq y \implies d(x,y) \geq c$] metric space with bounded geometry [$\forall R > 0$, $\sup_{x \in X} |B(x,R)| < \infty$].

The *translation algebra of* X, $\mathbb{C}[X]$, is the *-algebra of X-by-X matrices $(t_{yx})_{x,y \in X}$, $t_{yx} \in \mathbb{C}$, with *finite propagation* [there exists $R \geq 0$, so that $d(x,y) \geq R$ implies $t_{yx} = 0$] with uniformly bounded entries [$\sup_{x,y} |t_{xy}| < \infty$].



There is a *-representation $\lambda : \mathbb{C}[X] \to \mathscr{B}(\ell^2 X)$ "by multiplication". The *uniform Roe C*-algebra* $C^*_u X$ is the norm–closure of $\lambda(\mathbb{C}[X]) \subset \mathscr{B}(\ell^2 X)$.

If we replace "$\mathbb{C}$" by "$\mathscr{K}(H)$" (compact ops on ∞-dim'l separable Hilbert space H) above, and represent on $\ell^2(X,H)$, we get C^*X, the *Roe algebra* of X.

Coarse Baum–Connes conjecture

Slogan: The K-theory of Roe algebras serves as a receptacle for indices of (generalized) elliptic operators.

Coarse assembly map μ for a complete Riem. manifold M by example: If D is a "geometric" elliptic operator on M, then $\chi(D)$ is invertible modulo C^*M. So, the index $\mu(D)$ can be constructed in $K_0(C^*M)$.

Conjecture (The Coarse Baum–Connes conjecture)

For a metric space X with bounded geometry, the coarse assembly map

$$\mu : \lim_{d\to\infty} K_*(P_d X) \to K_*(C^*X)$$

is an isomorphism.

LHS: (Analytic) K-homology of the Rips complex of X. "Large–scale algebraic topology of X". "Geometric invariants of X".
RHS: "Algebraic topology of C^*X". Better properties of invariants (e.g. homotopy invariance).

Coarse Baum–Connes conjecture . . .

- Implies the Novikov conjecture (for groups). [Roe '95].
- True for spaces coarsely embeddable into a Hilbert space [Yu '00].
- False for certain expanders (e.g. box space of $SL_2(\mathbb{Z})$) [Higson '00].
- Open problem: does not CE(HSp) X imply X coarsely contains an expander?
- Injectivity proved for a large class of expanders (e.g. box spaces of f.g. linear groups); no counterexample [Guentner–Tessera–Yu '11]
- Fix? Replace C^*X by the "maximal version" $C^*_{\max}X$. Same K-theory as C^*X for CE(HSp) spaces [Willett–S '10]. The maximal version true for a large class of expanders [Oyono-Oyono–Yu '09, Willett–Yu '11, Chen–Wang–Yu '12]. Keywords: large girth, fibered coarse embedding into H.

Property A

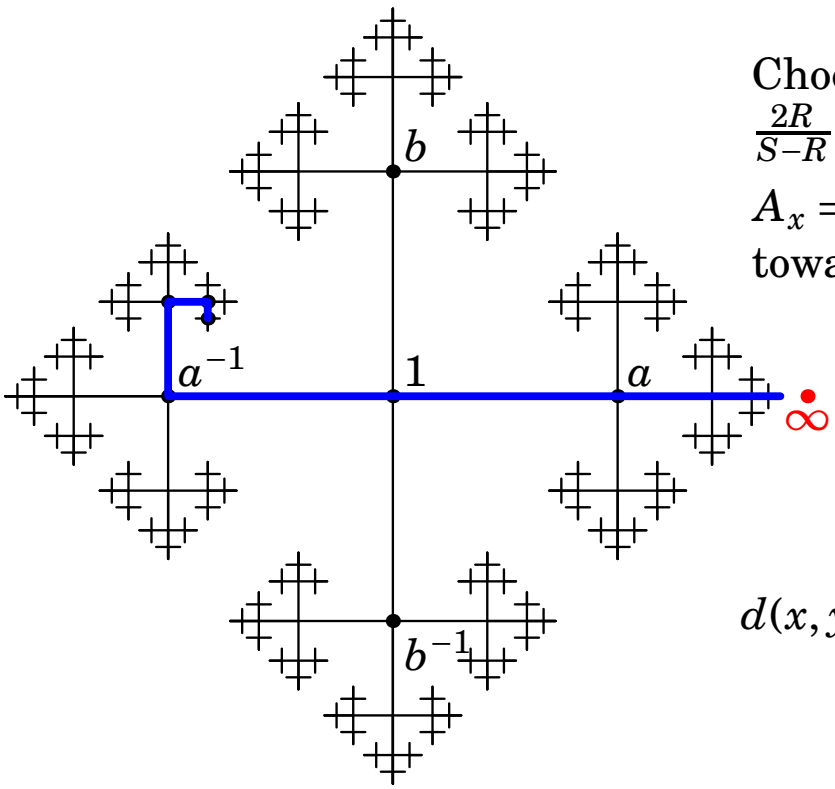
Definition (Yu '00)

X is said to have *property A*, if for every $R, \varepsilon > 0$ there exists $S \geq 0$ and finite subsets $A_x \subset X \times \mathbb{N}$ for each $x \in X$, such that

- $(x,1) \in A_x$ for every $x \in X$,
- $|A_x \triangle A_y| < \varepsilon |A_x \cap A_y|$ if $d(x,y) \leq R$ and
- the projection of A_x to X is contained in $B(x,S)$ for every $x \in X$.

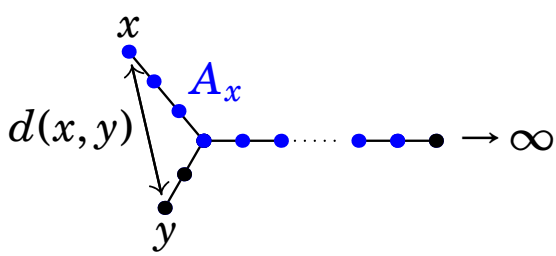
- "Non-equivariant amenability".
- Implies CE(HSp) (a criterion).
- Classes of discrete groups having A: amenable, hyperbolic, linear [Guentner–Higson–Weinberger '05], mapping class groups [Bestvina–Bromberg–Fujiwara '10].
- finite dim'l $CAT(0)$-cube complexes have A [Campbell–Niblo '04].
- Not known: Thompson's group F.
- What about *not having* A?

Free groups have A



Choose S>0 so that $\frac{2R}{S-R} < \varepsilon$.

$A_x = \{$ S points from x towards ∞ $\}$.



Property A and Others

equivariant side:		coarse side:
amenability	$\Longrightarrow$	property A
$\Downarrow$		$\Downarrow$
a-T-menability	$\Longrightarrow$	CE(HSp)

Theorem

For a finitely generated group Γ, the following are equivalent:

- Γ *has property A*
- Γ *acts amenably on some compact space [Higson–Roe '00]*
- $C^*_{\mathrm{red}}\Gamma$ *is an exact C*-algebra [Guentner–Kaminker, Ozawa '00]*
- $C^*_{\mathrm{u}}\Gamma$ *is a nuclear C*-algebra [Guentner–Kaminker, Ozawa '00]*

B exact means $\cdot \mapsto \cdot \otimes_{\min} B$ is exact. B nuclear means $B \otimes_{\min} \cdot = B \otimes_{\max} \cdot$.

Theorem (Willett–S '11)

If X has property A, then $C^*_{\mathrm{u}}X \cong C^*_{\mathrm{u}}Y$ implies $X \sim_c Y$.

Not Property A

Spaces of the sort $X = \sqcup_n X_n$; X_n finite graphs

- Expanders do not CE(HSp) [Gromov], so they do not have A.
- X does not have A if $\mathrm{girth}(X_n) \to \infty$ and degrees of vertices are between 3 and some $N < \infty$. [Willett '11]
- $\sqcup_n (\mathbb{Z}/2\mathbb{Z})^n$: not A, but CE(HSp). Not bounded geometry. [Nowak '07]
- A bdd. geom. example of X without A, but CE(HSp). [Arzhantseva–Guentner–S '10]

Non-exact Groups

- Gromov's Idea: take X as above and find a group with X in its Cayley graph. Done with some expanders [Arzhantseva–Delzant '09-12]. Tough. Small cancellation, hyperbolicity, randomness.
- Problem: Find an elementary construction. Find a non-A group which CE(HSp). [Arzhantseva...]: Find good labelings of graphs with large girth.

Uniform Local Amenability (ULA)

Definition (Følner)

A space X is *amenable*, if for all $R, \varepsilon > 0$ there exists finite $E \subset X$ with $|\partial_R E| < \varepsilon |E|$. $[\partial_R E = B_R(E) \setminus E]$

- ULA is a coarse invariant.
- A implies ULA.
- Main advantage: "easy" to check that it fails (expanders, families with large girth).
- "Localizing with finite measures instead of sets": $\mathrm{ULA}_\mu \iff$ A $\iff$ MSP $\iff$ ONL, for bounded geometry X. [BNSWW+Sako '12]
- Open problem: relation between CE(HSp) and ULA? (Known CE: $\not\Rightarrow$ ULA: [AGS] example.)