COARSE GEOMETRY AND ROE C*-ALGEBRAS

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OUTLINE

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2 ROE ALGEBRAS

3 PROPERTY A AND COARSE EMBEDDINGS
**OBJECTS OF INTEREST**

Slogan: Study spaces from a large–scale perspective.

Notation: $X, Y \ldots$ metric spaces; $d$ metric. $\Gamma$ discrete group.

Examples:

- Finitely generated groups with word metric: $\Gamma = \langle S \rangle$, $S = S^{-1}$, $|S| < \infty$. Define a metric by $d_S(g, h) =$ the length of a shortest word in alphabet $S$ representing $g^{-1}h$.
  
  E.g. if $\mathbb{Z} = \langle 1, -1 \rangle$, then the metric is $d(m, n) = |n - m|$.

- Graphs (finite or infinite), endowed with the path metric.

- Complete Riemannian manifolds.

- “Coarse disjoint union” $X = \bigsqcup_n G_n$ of a sequence of finite graphs $(G_n)$. Metric: on each $G_n$ the path metric, $d(G_n, G_m) = m + n + |G_n| + |G_m|$.

- “Box space”: $X = \bigsqcup_n \text{Cayley}(\Gamma/\Gamma_n; S/\Gamma_n)$, where $\Gamma = \langle S \rangle$, $|S| < \infty$, $\Gamma_n \leq \Gamma$ normal, $[\Gamma : \Gamma_n] < \infty$.

**EXPANDERS**

For a finite graph $G$, the Cheeger constant is

$$h(G) = \min \left\{ \frac{|\partial S|}{|S|} : S \subset V(G), 0 < |S| \leq \frac{|G_n|}{2} \right\}.$$

A sequence of expanders (expander) is a sequence of finite graphs $G_n$, such that

- the degrees of vertices are uniformly bounded,
- $|G_n| \not\to \infty$ and
- $\inf_n h(G_n) > 0$.

Think of it as a metric space $X = \bigsqcup_n G_n$.

First examples: Box spaces of residually finite groups with property (T), e.g. $\bigsqcup_p SL_n(\mathbb{Z})/SL_n(\mathbb{Z}/p\mathbb{Z})$. [Margulis]

In fact, a group $\Gamma$ has property $(\tau)$ with respect to a family of finite index subgroups $(\Gamma_n)_{n \in \mathbb{N}}$ iff the box space $X = \bigsqcup_n (\Gamma/\Gamma_n)$ is an expander.
COARSE DEFINITIONS

A map $f : X \to Z$ is coarse, if

- $\exists \rho_+ : [0, \infty) \to [0, \infty)$, such that $d(f(x), f(y)) \leq \rho_+(d(x, y))$ for all $x, y \in X$;
- $|f^{-1}(z)| < \infty$ for all $z \in Z$.

Maps $f, g : X \to Z$ are close, if $\sup \{d(f(x), g(x)) \mid x \in X\} < \infty$.

Spaces $X$ and $Z$ are coarsely equivalent, written $X \sim_c Z$, if there exist coarse $f : X \to Z$, $g : Z \to X$, with $f \circ g$ is close to $\text{id}_Z$ and $g \circ f$ is close to $\text{id}_X$.

A map $f : X \to Z$ is a ($\star$) embedding, if there exist $\rho_+, \rho_- : [0, \infty) \to [0, \infty)$ with ($\star \star$), such that

$$\rho_-(d(x, y)) \leq d(f(x), f(y)) \leq \rho_+(d(x, y)).$$

<table>
<thead>
<tr>
<th>coarse (CE)</th>
<th>($\star \star$)</th>
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<tbody>
<tr>
<td>quasi–isometric (QI)</td>
<td>$\rho_- \nearrow \infty$</td>
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<tr>
<td>bilipschitz</td>
<td>$\rho_+$ and $\rho_-$ are affine ($Ax + B$)</td>
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<tr>
<td></td>
<td>$\rho_+$ and $\rho_-$ are linear ($Ax$)</td>
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A ($\star$) embedding $f$ is a ($\star$) equivalence, if $\sup \{d(z, f(X)) \mid z \in Z\} < \infty$.

EXAMPLES

EQUIVALENCES:

- Bounded space $\sim_c \{\text{pt}\}$.
- $\mathbb{Z}^n \sim_c \mathbb{R}^n$, but $\mathbb{Z}^m \sim_c \mathbb{Z}^n$ $\implies$ $m = n$ (asymptotic dimension).
- Free groups $F_r$ (with the free generating sets), $2 \leq r < \infty$ are all bilipschitz equivalent [Papasoglu '95].
- $X = \{2^{2n} \mid n \in \mathbb{N}\} \subset \mathbb{N}$, metric from $\mathbb{N}$. Any bijection is a coarse equivalence, but any QI $X \to X$ is eventually constant.

BASIC GEOMETRIC GROUP THEORY LEMMA:
If $\Gamma = \langle S \rangle = \langle S' \rangle$, $|S|, |S'| < \infty$, then $(\Gamma, d_S) \sim_{QI} (\Gamma, d_{S'})$. So any QI invariant is actually an invariant of the underlying group.

ŠVARC–MILNOR THEOREM:
If $\Gamma$ acts properly and cocompactly on a length space $X$, then $\Gamma \sim_{QI} X$.
[So $\pi_1(M) \sim_{QI} \tilde{M}$ for a compact Riemannian manifold $M$.]
Coarse Properties and Some Theorems

... amenability, asymptotic dimension, coarse embeddability into ___ (e.g. a Hilbert space)

(Gromov) hyperbolicity is a QI invariant of geodesic spaces.

**Theorem (G. Yu ’97)**

*If $M$ is a uniformly contractible complete Riemannian manifold with bounded geometry and finite asymptotic dimension, then it admits no metric of uniformly positive scalar curvature, within the class of CE metrics.*

**Theorem (G. Yu ’00)**

*If $\Gamma$ admits a coarse embedding into a Hilbert space, then the Novikov conjecture holds for $\Gamma$.***

Definitions of Roe Algebras

Slogan: Roe algebras are $C^*$-algebras which encode the coarse structure of a space.

Let $X$ be a uniformly discrete [$\exists c > 0 \text{ with } x \neq y \implies d(x, y) \geq c$] metric space with bounded geometry [$\forall R > 0$, $\sup_{x \in X} |B(x, R)| < \infty$].

The *translation algebra of $X$, $\mathbb{C}[X]$, is the $*$-algebra of $X$-by-$X$ matrices $(t_{xy})_{x,y \in X}$, $t_{xy} \in \mathbb{C}$, with finite propagation [there exists $R \geq 0$, so that $d(x, y) \geq R$ implies $t_{xy} = 0$] with uniformly bounded entries [$\sup_{x,y} |t_{xy}| < \infty$].

There is a $*$-representation $\lambda : \mathbb{C}[X] \to \mathcal{B}(\ell^2 X)$ “by multiplication”. The uniform Roe $C^*$-algebra $C_u^*X$ is the norm-closure of $\lambda(\mathbb{C}[X]) \subset \mathcal{B}(\ell^2 X)$.

If we replace “$\mathbb{C}$” by “$\mathcal{K}(H)$” (compact ops on $\infty$-dim’l separable Hilbert space $H$) above, and represent on $\ell^2(X, H)$, we get $C^*X$, the Roe algebra of $X$. 
**Coarse Baum–Connes conjecture**

Slogan: The $K$-theory of Roe algebras serves as a receptacle for indices of (generalized) elliptic operators.

*Coarse assembly map* $\mu$ for a complete Riem. manifold $M$ by example: If $D$ is a “geometric” elliptic operator on $M$, then $\chi(D)$ is invertible modulo $C^*M$. So, the index $\mu(D)$ can be constructed in $K_0(C^*M)$.

**Conjecture (The Coarse Baum–Connes conjecture)**

For a metric space $X$ with bounded geometry, the coarse assembly map

$$\mu : \lim_{d \to \infty} K_*(P_dX) \to K_*(C^*X)$$

is an isomorphism.


RHS: “Algebraic topology of $C^*X$”. Better properties of invariants (e.g. homotopy invariance).

• Implies the Novikov conjecture (for groups). [Roe ’95].
• True for spaces coarsely embeddable into a Hilbert space [Yu ’00].
• False for certain expanders (e.g. box space of $SL_2(\mathbb{Z})$) [Higson ’00].
• Open problem: does not CE(HSp) $X$ imply $X$ coarsely contains an expander?
• Injectivity proved for a large class of expanders (e.g. box spaces of f.g. linear groups); no counterexample [Guentner–Tessera–Yu ’11]
• Fix? Replace $C^*X$ by the “maximal version” $C^*_mX$. Same $K$-theory as $C^*X$ for CE(HSp) spaces [Willett–S ’10]. The maximal version true for a large class of expanders [Oyono-Oyono–Yu ’09, Willett–Yu ’11, Chen–Wang–Yu ’12]. Keywords: large girth, fibered coarse embedding into $H$. 
**Property A**

**Definition (Yu ’00)**

X is said to have property A, if for every $R, \varepsilon > 0$ there exists $S \geq 0$ and finite subsets $A_x \subset X \times \mathbb{N}$ for each $x \in X$, such that

- $(x, 1) \in A_x$ for every $x \in X$,
- $|A_x \triangle A_y| < \varepsilon |A_x \cap A_y|$ if $d(x, y) \leq R$ and
- the projection of $A_x$ to $X$ is contained in $B(x, S)$ for every $x \in X$.

- “Non-equivariant amenability”.
- Implies CE(HSp) (a criterion).
- Classes of discrete groups having A: amenable, hyperbolic, linear [Guentner–Higson–Weinberger ’05], mapping class groups [Bestvina–Bromberg–Fujiwara ’10].
- finite dim’l $CAT(0)$-cube complexes have A [Campbell–Niblo ’04].
- Not known: Thompson’s group $F$.
- What about not having A?

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**Free Groups Have A**

Choose $S > 0$ so that $\frac{2R}{S-R} < \varepsilon$.

$A_x = \{ S \text{ points from } x \text{ towards } \infty \}$. 

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**Ján Špakula ( Uni Münster ) Coarse geometry and Roe C*-algebras Oct 25, 2012 15 / 18**
**Property A and Others**

<table>
<thead>
<tr>
<th>equivariant side:</th>
<th>coarse side:</th>
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<tr>
<td>amenability</td>
<td>property A</td>
</tr>
<tr>
<td>a-T-menability</td>
<td>CE(HSp)</td>
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**Theorem**

For a finitely generated group $\Gamma$, the following are equivalent:

- $\Gamma$ has property A
- $\Gamma$ acts amenably on some compact space [Higson–Roe ’00]
- $C^*_r \Gamma$ is an exact C*-algebra [Guentner–Kaminker, Ozawa ’00]
- $C^*_u \Gamma$ is a nuclear C*-algebra [Guentner–Kaminker, Ozawa ’00]

B exact means $\cdot \rightarrow \otimes_{\text{min}} B$ is exact. B nuclear means $B \otimes_{\text{min}} \cdot = B \otimes_{\text{max}} \cdot$.

**Theorem (Willett–S ’11)**

If $X$ has property A, then $C^*_u X \cong C^*_u Y$ implies $X \sim_c Y$.

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**Not Property A**

**Spaces** of the sort $X = \bigsqcup_n X_n; X_n$ finite graphs

- Expanders do not CE(HSp) [Gromov], so they do not have A.
- $X$ does not have A if girth($X_n$) $\rightarrow \infty$ and degrees of vertices are between 3 and some $N < \infty$. [Willett ’11]
- $\bigsqcup_n (\mathbb{Z}/2\mathbb{Z})^n$: not A, but CE(HSp). Not bounded geometry. [Nowak ’07]
- A bdd. geom. example of $X$ without A, but CE(HSp). [Arzhantseva–Guentner–S ’10]

**Non-exact Groups**

- Gromov’s Idea: take $X$ as above and find a group with $X$ in its Cayley graph. Done with some expanders [Arzhantseva–Delzant ’09-12]. Tough. Small cancellation, hyperbolicity, randomness.
- Problem: Find an elementary construction. Find a non-A group which CE(HSp). [Arzhantseva...]: Find good labelings of graphs with large girth.
Uniform Local Amenability (ULA)

**Definition (Følner)**

A space $X$ is *amenable*, if for all $R, \varepsilon > 0$ there exists finite $E \subset X$ with

$$|\partial^*_R E| < \varepsilon |E|$$

and

$$[\partial^*_R E = B_R(E) \setminus E]$$

- ULA is a coarse invariant.
- A implies ULA.
- Main advantage: “easy” to check that it fails (expanders, families with large girth).
- “Localizing with finite measures instead of sets”: $\text{ULA}_\mu \iff A \iff \text{MSP} \iff \text{ONL}$, for bounded geometry $X$. [BNSWW+Sako ’12]
- Open problem: relation between CE(HSp) and ULA? (Known CE: $\not\Rightarrow ULA$: [AGS] example.)